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Evolution of twist-three parton distributions in QCD beyond the large-N_c limit

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Abstract:

We formulate a consistent $1/N_c^2$ expansion of the QCD evolution equations for the twist-three quark distributions $g_2(x,Q^2)$, $h_L(x,Q^2)$ and $e(x,Q^2)$ based on the interpretation of the evolution as a three-particle quantum-mechanical problem with hermitian Hamiltonian. Each distribution amplitude can be decomposed in contributions of partonic components with DGLAP-type scale dependence. We calculate the $1/N_c^2$ corrections to the evolution of the dominant component with the lowest anomalous dimension – the only one that survives in the large-N_c limit – and observe a good agreement with the exact numerical results for $N_c = 3$. The $1/N_c^2$ admixture of operators with higher anomalous dimensions is shown to be concentrated at a few lowest partonic components and in general is rather weak.

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1. Twist-three parton distributions in the nucleon are attracting increasing interest as unique probes of novel quark-gluon correlations in hadrons with clear experimental signature, giving rise to certain asymmetries in experiments with polarized beams and targets. Quantitative studies of such asymmetries are becoming possible with the increasing precision of experimental data at SLAC and RHIC, and can provide for an important part of the future spin physics program on high-luminosity accelerators like ELFE, upgraded CEBAF etc. With this perspective, a detailed theoretical study of twist-three parton distributions in QCD becomes mandatory.

Altogether, there exist three twist-3 distribution functions [1] – chiral-odd, $e(x,Q^2)$ and $h_L(x,Q^2)$, and chiral-even, $g_2(x,Q^2)$ – each being a function of parton momentum fraction x and the energy scale Q^2. By virtue of Lorentz invariance the distributions h_L and g_2 contain contributions of twist-2 structure functions h_1 and g_1, respectively:

$$h_L(x) = 2x\int_x^1 \frac{dy}{y^2}h_1(y) + \widetilde{h}_L(x)\,, \qquad g_2(x) = -g_1(x) + \int_x^1 \frac{dy}{y}g_1(y) + \widetilde{g}_2(x)\,. \tag{1}$$

The QCD description of the remaining genuine twist-3 part of these distributions functions, $\widetilde{h}_L(x)$, $\widetilde{g}_2(x)$ and of $e(x)$, is usually believed to be quite sophisticated. Their moments are related through the QCD equations of motion to the matrix elements of quark-antiquark-gluon operators which have a nontrivial scale dependence and mix with each other under the renormalization.† As a consequence, the QCD evolution equations for the functions $\widetilde{h}_L(x)$, $\widetilde{g}_2(x)$, $e(x)$ cannot be written in a closed form and require additional nonperturbative input. An important simplification occurs, however, in the large-N_c limit. It was shown [2, 3] that to this accuracy the twist-three distributions $\widetilde{h}_L$ and e as well as the flavor-nonsinglet contribution to $\widetilde{g}_2$ satisfy simple DGLAP-type evolution equations (see e.g. [4])

$$\begin{aligned}
&Q^2\frac{d}{dQ^2}f(x,Q^2) = \frac{\alpha_s}{4\pi}\int_x^1 \frac{dz}{z}P_f^{(0)}(x/z)f(z,Q^2)\,, \qquad f = \{\widetilde{h}_L\,, \widetilde{g}_2^{\rm NS}\,, e\}\,,\\
&P_f^{(0)}(z) = 2N_c\left\{\left[\frac{1}{1-z}\right]_+ + \frac{1}{4}\,\delta(1-z) + \sigma_f - \frac{1}{2}\right\},
\end{aligned} \tag{2}$$

where $\sigma_f = -1\,, 0$ and 1 for $f = \widetilde{h}_L, \widetilde{g}_2$ and e, respectively, and $1/(1-z)_+ = 1/(1-z) - \delta(1-z)\int_0^1 dz'/(1-z')$. This result implies that the inclusive measurements of twist-three distributions are complete (to the stated accuracy) in the sense that knowledge of the distribution at one value of Q^2 is enough to predict the distribution at arbitrary Q^2. From phenomenological point of view it allows to relate the measurements of different experiments to each other and to compare them to the model predictions (including lattice calculations) that typically refer to a low scale.

Although the large-N_c version of QCD presents a valid theoretical limit, its relevance to the actual $N_c = 3$ world and its accuracy in predicting the scale dependence of twist-3

†Additional mixture with three-gluon operators is present for flavor-singlet contribution to $g_2(x,Q^2)$. In this letter we consider only the flavor-nonsinglet part.

distributions is *a priory* not clear. This work presents the first attempt to go beyond the large-N_c approximation in a systematic way. In particular, we calculate the $1/N_c^2$ corrections to the evolution kernels in (2) and show that mixing with quark-antiquark-gluon operators remains under control.

2. From the OPE analysis [5] one finds that the scale dependence of the moments of the twist-3 distributions $\int_{-1}^{1} dx x^{N+2} f(x)$ is governed by renormalization of the set of local composite quark-antiquark-gluon operators ($k = 0, ..., N$)

$$
\begin{aligned}
{[S^{\pm}_{\mu}]}^k_N &= \bar{q}(\overleftarrow{D}\cdot n)^k \not{n}\, n^{\nu}[\widetilde{G}_{\mu\nu} \pm iG_{\mu\nu}\gamma_5]\,(\overrightarrow{D}\cdot n)^{N-k} q\,, \\
{[T_{\Gamma}]}^k_N &= \bar{q}(\overleftarrow{D}\cdot n)^k n_{\mu}\sigma^{\mu\rho}\,\Gamma\, n^{\nu} G_{\nu\rho}\,(\overrightarrow{D}\cdot n)^{N-k} q\,, \qquad \Gamma = \{\mathbb{1},\, i\gamma_5\}
\end{aligned}
\tag{3}
$$

for chiral-even ($S^{\pm}_{\mu}$) and chiral-odd distributions (T_{I} and $T_{i\gamma_5}$ for $e(x)$ and $\widetilde{h}_L(x)$, respectively), see e.g. [6]. Here, n_μ is a light-like vector and $\widetilde{G}_{\mu\nu} = \epsilon_{\mu\nu\rho\lambda}G^{\rho\lambda}/2$ stands for a dual gluon field strength. To leading order, renormalization of T_{I} and $T_{i\gamma_5}$ is the same and we, therefore, drop the subscript in what follows. Similarly, it is enough to consider the operator S^+.

The operators $[T]^k_N$ (and $[S^+]^k_N$) with different $k = 0, ..., N$ and the same number of covariant derivatives $N \geq 0$ mix with each other under renormalization. The mixing matrices have been calculated to the leading order (e.g. [6]) and can be diagonalized numerically for any given N. The eigenvectors, then, define the multiplicatively renormalizable operators and the eigenvalues give their anomalous dimensions. A disadvantage of this (traditional) approach is that the mixing matrix does not have any obvious structure in this basis and is not symmetric. As the result, the structure of the spectrum remains obscure and the eigenvectors are not mutually orthogonal with any simple weight function. Relative importance of various contributions is also far from being clear.[‡]

A systematic $1/N_c^2$ expansion of the evolution equations is made possible by going over to the Hamiltonian formulation developed in in [7, 8, 9, 10, 11]. The renormalization group evolution is driven to leading logarithmic accuracy by tree-level counterterms and has the conformal symmetry of the QCD Lagrangian. The evolution kernel can, therefore, be written in an abstract operator form in terms of Casimir operators of the collinear subgroup $SL(2, R)$ of the conformal group. In this way, diagonalization of the mixing matrix of the twist-3 operators in (3) can be reformulated as a three-particle quantum mechanical problem

$$
\mathcal{H}\,\Psi_{N,q}(x_1, x_2, x_3) = \mathcal{E}_{N,q}\Psi_{N,q}(x_1, x_2, x_3), \tag{4}
$$

defined by the Hamiltonian [7, 8, 10, 11]

$$
\mathcal{H}_A = N_c\mathcal{H}^{(0)}_A - \frac{2}{N_c}\mathcal{H}^{(1)}_A\,, \qquad A = \{T\,, S^+\}\,, \tag{5}
$$

[‡]E.g. the hierarchy of entries in the mixing matrix $\mathcal{O}(N)$, $\mathcal{O}(1)$, $\mathcal{O}(1/N)$, etc. is not preserved in the eigenvalues.

with

$$\mathcal{H}_T^{(0)} = V_{qg}^{(0)}(J_{12}) + V_{qg}^{(0)}(J_{23})\,, \qquad \mathcal{H}_T^{(1)} = V_{qg}^{(1)}(J_{12}) + V_{qg}^{(1)}(J_{23}) + V_{qq}^{(1)}(J_{13})\,,$$
$$\mathcal{H}_{S+}^{(0)} = V_{qg}^{(0)}(J_{12}) + U_{qg}^{(0)}(J_{23})\,, \qquad \mathcal{H}_{S+}^{(1)} = V_{qg}^{(1)}(J_{12}) + U_{qg}^{(1)}(J_{23}) + U_{qq}^{(1)}(J_{13})\,. \tag{6}$$

Here, the notation was introduced for two-particle quark-quark and quark-gluon kernels

$$V_{qg}^{(0)}(J) = \psi(J+3/2) + \psi(J-3/2) - 2\psi(1) - 3/4\,, \tag{7}$$
$$U_{qg}^{(0)}(J) = \psi(J+1/2) + \psi(J-1/2) - 2\psi(1) - 3/4\,,$$
$$V_{qg}^{(1)}(J) = \frac{(-1)^{J-5/2}}{(J-3/2)(J-1/2)(J+1/2)}\,, \quad U_{qg}^{(1)}(J) = -\frac{(-1)^{J-5/2}}{2(J-1/2)}\,,$$
$$V_{qq}^{(1)}(J) = \psi(J) - \psi(1) - 3/4\,, \quad U_{qq}^{(1)}(J) = \frac{1}{2}\left[\psi(J-1) + \psi(J+1)\right] - \psi(1) - 3/4\,.$$

Here and below $\psi(x) = d\ln\Gamma(x)/dx$ stands for the Euler Ψ-function; the subscripts '1,2,3' refer to antiquark, gluon and quark fields, respectively. The operators J_{ik} are defined as follows

$$J_{ik}\,(J_{ik}-1) = L_{ik}^2 = (\vec{L}_i + \vec{L}_k)^2\,, \tag{8}$$

where $\vec{L}_i$ are the generators of the $SL(2,R)$ group and L_{ik}^2 are the corresponding two-particle Casimir operators. The generators $\vec{L}_i$ can be realized as differential operators acting on coordinates x_i of the wave function $\Psi(x_1,x_2,x_3)$

$$L_{-,i} = x_i\partial_{x_i}^2 + 2j_i\partial_{x_i}\,, \qquad L_{+,i} = -x_i\,, \qquad L_{0,i} = x_i\partial_{x_i} + j_i\,, \tag{9}$$

where the conformal spin j is equal to $j_1 = j_3 = 1$ for the (anti)quark and $j_2 = 3/2$ for the gluon field, respectively. The eigenvalues of the operator J_{ik} define the possible values of the conformal spin in the two-parton channel and are given by $j_i + j_k + n$ with n being nonnegative integer.

The Hamiltonians defined above are manifestly $SL(2,R)$ invariant:

$$[\mathcal{H}, L_\alpha] = [\mathcal{H}, L^2] = [L^2, L_\alpha] = 0, \tag{10}$$

where L_α $(\alpha = 0,+,-)$ are the total three-particle $SL(2)$ generators

$$L_\alpha = L_{\alpha,1} + L_{\alpha,2} + L_{\alpha,3}\,, \qquad L^2 = L_0(L_0-1) + L_+L_-\,. \tag{11}$$

One can, therefore, diagonalize the three operators L^2, L_0 and $\mathcal{H}$ simultaneously. The conditions

$$L_0\,\Psi_{N,q}(x_1,x_2,x_3) = (N+7/2)\,\Psi_{N,q}(x_1,x_2,x_3)\,, \qquad L_-\,\Psi_{N,q}(x_1,x_2,x_3) = 0 \tag{12}$$

define $\Psi_{N,q}(x_1,x_2,x_3)$ to be a homogeneous polynomial of degree N that does not contain factors of $(x_1+x_2+x_3)$ and therefore does not vanish as $\sum_i x_i = 0$.

Solving the Schrödinger equation (4) one constructs the basis of multiplicatively renormalizable local operators, $\mathcal{O}_{N,q}$. Omitting the Lorentz structure, the correspondence is, schematically [9]

$$\mathcal{O}_{N,q} = \Psi_{N,q}(D_{\bar{q}}, D_g, D_q)\, \bar{q}\, G\, q \tag{13}$$

with covariant derivatives $D_{\bar{q}}, D_g, D_q$ acting on the antiquark, gluon and quark, respectively. The corresponding eigenvalues provide the anomalous dimensions $\mathcal{E}_{N,q}$:

$$\mathcal{O}_{N,q}(Q^2) = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{\mathcal{E}_{N,q}/b} \mathcal{O}_{N,q}(\mu^2)\,, \tag{14}$$

where $b = 11N_c/3 - 2n_f/3$. Here, parameter q enumerates different eigenstates of the mixing matrix.

Note that the conformal operators defined in (13) diagonalize the full mixing matrix, including mixing with the operators containing total derivatives. Since at the end only forward matrix elements of these operators enter the parton distributions, taking into account this additional mixing may be seen as an unnecessary complication. The advantage for such formulation is, however, that the Hamiltonian becomes hermitian with respect to the scalar product

$$\langle \Psi_1 | \Psi_2 \rangle = \int_0^1 \mathcal{D}x\, x_1 x_2^2 x_3 \Psi_1^*(x_i) \Psi_2(x_i) \tag{15}$$

with $\mathcal{D}x = dx_1 dx_2 dx_3 \delta(x_1{+}x_2{+}x_3{-}1)$. As well known in quantum mechanics, hermiticity of the Hamiltonian implies that all its eigenvalues (anomalous dimensions) $\mathcal{E}_{N,q}$ are real[§], and the eigenfunctions are mutually orthogonal:

$$\langle \Psi_{N,q} | \Psi_{N',q'} \rangle = \delta_{NN'}\delta_{qq'} \|\Psi_{N,q}\|^2, \qquad \|\Psi_{N,q}\|^2 = \int_0^1 \mathcal{D}x\, x_1 x_2^2 x_3 |\Psi_{N,q}(x_i)|^2\,. \tag{16}$$

Completeness and orthogonality of the eigenstates corresponding to multiplicatively renormalizable operators will be of crucial importance for this work. In particular, using these properties one can expand the moments of the twist-3 parton distributions over the contributions of multiplicatively renormalizable operators as

$$\int_{-1}^1 dx\, x^{N+2} f(x, Q^2) = \sum_q \frac{\langle \Psi_{N,q} | \Phi_N \rangle}{\|\Psi_{N,q}\|^2} \langle\!\langle \mathcal{O}_{N,q}(Q^2) \rangle\!\rangle\,. \tag{17}$$

Here, $\langle\!\langle \mathcal{O}_{N,q} \rangle\!\rangle$ are reduced (scalar and dimensionless) forward matrix elements of the corresponding local operators over the nucleon state and $\Phi_N(x_i)$ are the coefficients in the

[§] Although the conformal symmetry is lost beyond one-loop, it is easy to prove that anomalous dimensions of twist-3 operators remain real to all orders in perturbation theory. Indeed, entries in the mixing matrix are real and complex eigenvalues may only appear in (complex conjugate) pairs. On the other hand, the number of eigenvalues is fixed and in leading order they are non-degenerate (which one can establish by inspection). These two conditions are contradictory since a pair of complex conjugate eigenvalues (i.e. with the same real parts) cannot be obtained from a discrete non-degenerate spectrum at $\alpha_s \to 0$ (i.e. with all real parts different) by a perturbative renormalization group flow.

re-expansion of twist-3 quark-antiquark operators that define the parton distributions, in terms of quark-antiquark-gluon operators. They are given by QCD equations of motion and for $x_1 + x_2 + x_3 = 0$ take the simple form [11]

$$\Phi_N^{T,+} = \frac{x_1^{N+1} - (-x_3)^{N+1}}{x_1 + x_3}\,, \quad \Phi_N^{T,-} = (\partial_{x_1} + \partial_{x_3})\frac{\Phi_{N+1}^{T,+}}{N+2}\,, \quad \Phi_N^{S^+} = \partial_{x_1}\frac{\Phi_{N+1}^{T,+}}{N+2} \tag{18}$$

for $f = e\,, \widetilde{h}_L$ and $\widetilde{g}_2$, respectively. Note, however, that in order to form the scalar product in (17) one needs to know $\Phi_N(x_i)$ for $x_1 + x_2 + x_3 = 1$ that corresponds to taking into account contributions to the QCD equations of motion of the operators containing total derivatives. The corresponding expressions are given below in (31). Each term in (17) has an autonomous Q^2-evolution, Eq. (14), and brings a new nonperturbative parameter $\langle\!\langle \mathcal{O}_{N,q}(\mu^2)\rangle\!\rangle$.

3. The large-N_c Hamiltonians $\mathcal{H}^{(0)}$ in (6) possess an additional 'hidden' symmetry, related to the existence of a nontrivial conserved charge [8]

$$[\mathcal{H}_T^{(0)}, Q_T] = 0\,, \qquad Q_T = \{L_{12}^2, L_{23}^2\} - \frac{9}{2}L_{12}^2 - \frac{9}{2}L_{23}^2\,,$$

$$[\mathcal{H}_{S^+}^{(0)}, Q_{S^+}] = 0\,, \qquad Q_{S^+} = \{L_{12}^2, L_{23}^2\} - \frac{1}{2}L_{12}^2 - \frac{9}{2}L_{23}^2\,, \tag{19}$$

where $\{\ ,\ \}$ stands for the anticommutator. As the result, eigenstates of $\mathcal{H}^{(0)}$ can be labeled by quantized eigenvalues q_ℓ, $(\ell = 0, 1, \ldots, N)$, of the operator Q:

$$Q\,\Psi_{N,q}^{(0)}(x_1, x_2, x_3) = q\,\Psi_{N,q}^{(0)}(x_1, x_2, x_3)\,. \tag{20}$$

Since the number of degrees of freedom (=3) equals in this case to the number of conserved charges (L^2, L_0, Q), the quantum mechanical system described by the Hamiltonian $\mathcal{H}^{(0)}$ is completely integrable. This allows to use advanced mathematical methods for the analysis of the spectrum and, in particular, develop the WKB expansion of the eigenvalues at large N [8, 10, 11].

The remarkable property of the large-N_c evolution kernels [2, 3] that is responsible for the simplicity of the evolution in (2) is that the eigenfunctions corresponding to the states with lowest energy coincide with the coefficient functions Φ_N entering the expansion in Eq. (17) [10, 11]

$$\Phi_N(x_i) = \Psi_{N,q_0}^{(0)}(x_i)\,, \quad q_0^S = (N+3)^2 + 3/8\,, \quad q_0^{T,\pm} = (N+3\mp 2)^2 - 53/8\,. \tag{21}$$

To be precise, $\Phi_N^{S^+}(x_i)$ coincides with the ground state of $\mathcal{H}_{S^+}^{(0)}$ and $\Phi_N^{T,\pm}(x_i)$ with the two lowest eigenstates of $\mathcal{H}_T^{(0)}$ with opposite parity with respect to permutations of quarks, $x_1 \rightleftarrows x_3$. The corresponding eigenvalues determine the lowest anomalous dimensions for each N, $\min_q \mathcal{E}_{N,q} = \mathcal{E}_{N,q_0} = N_c E_N + \mathcal{O}(1/N_c)$ and they are equal to

$$E_N^{T,\pm} = 2\psi(N+3) + \frac{1\mp 2}{N+3} - \frac{1}{2} + 2\gamma_{\rm E}\,,$$

$$E_N^S = 2\psi(N+3) + \frac{1}{N+3} - \frac{1}{2} + 2\gamma_{\rm E}\,. \tag{22}$$

We will often omit the subscript "$q = q_0$" for these special states. Note useful relations:

$$2E_N^S = E_N^{T,+} + E_N^{T,-}\,, \quad 2\Phi_N^{S\pm}(x_i) = \Phi_N^{T,+}(x_i) \pm \Phi_N^{T,-}(x_i)\,, \quad \langle \Phi_N^{T,-} | \Phi_N^{T,+} \rangle = 0\,. \tag{23}$$

Since the wave functions of eigenstates with different energies are mutually orthogonal, Eq. (21) implies that the single term $q = q_0$ survives in the sum (17) in the leading large N_c limit. As a consequence, the moments of twist-3 parton distributions are entirely given by the reduced matrix elements of the corresponding local operators $\mathcal{O}_{N,q_0}$, Eq. (13) [2, 3]:

$$\begin{aligned}
\int_{-1}^{1} dx\, x^{N+2}\, e(x,Q^2) &= \langle\!\langle \mathcal{O}_N^{T,+}(Q^2) \rangle\!\rangle\,, \\
(N+4)\int_{-1}^{1} dx\, x^{N+2}\, \widetilde{h}_L(x,Q^2) &= \langle\!\langle \mathcal{O}_N^{T,-}(Q^2) \rangle\!\rangle\,, \\
\frac{N+3}{N+2}\int_{-1}^{1} dx\, x^{N+2}\, \widetilde{g}_2(x,Q^2) &= \langle\!\langle \mathcal{O}_N^{S^+}(Q^2) \rangle\!\rangle + \langle\!\langle \mathcal{O}_N^{S^-}(Q^2) \rangle\!\rangle\,.
\end{aligned} \tag{24}$$

Combined with the scale dependence of matrix elements, Eq. (14), these relations are equivalent to the DGLAP evolution equations in (2) with $N_c E_N^f = -\int_0^1 dz\, z^{N+2} P_f^{(0)}(z)$.

Treating the $\sim 1/N_c$ contribution to the Hamiltonian in (5) as a perturbation, we expand

$$\begin{aligned}
\mathcal{E}_{N,q} &= N_c E_{N,q} + N_c^{-1}\delta E_{N,q} + \dots\,, \\
\Psi_{N,q} &= \Psi_{N,q}^{(0)} + N_c^{-2}\delta\Psi_{N,q} + \dots\,,
\end{aligned} \tag{25}$$

with the usual quantum-mechanical expressions

$$\delta E_{N,q} = -2\|\Psi_{N,q}^{(0)}\|^{-2}\, \langle \Psi_{N,q}^{(0)} | \mathcal{H}^{(1)} | \Psi_{N,q}^{(0)} \rangle\,, \tag{26}$$

$$\delta\Psi_{N,q}(x_i) = -2\sum_{q'\neq q} \frac{\langle \Psi_{N,q}^{(0)} | \mathcal{H}^{(1)} | \Psi_{N,q'}^{(0)} \rangle}{\|\Psi_{N,q'}^{(0)}\|^2}\, \frac{\Psi_{N,q'}^{(0)}(x_i)}{E_{N,q} - E_{N,q'}}\,. \tag{27}$$

To this accuracy, moments of the twist-3 distributions are no longer renormalized multiplicatively and have to be expanded in contributions of multiplicatively renormalizable operators, Eq. (17):

$$c_f(N)\int_{-1}^{1} dx\, x^{N+2}\, f(x,Q^2) = \langle\!\langle \mathcal{O}_{N,q_0}(Q^2) \rangle\!\rangle - \frac{2}{N_c^2}\sum_{q\neq q_0} \frac{\langle \Psi_{N,q}^{(0)} | \mathcal{H}^{(1)} | \Phi_N \rangle}{\|\Psi_{N,q}^{(0)}\|^2} \cdot \frac{\langle\!\langle \mathcal{O}_{N,q}(Q^2) \rangle\!\rangle}{E_{N,q} - E_{N,q_0}}\,, \tag{28}$$

where $c_e = 1$, $c_{h_L} = N+4$, $c_{g_2} = (N+3)/(N+2)$, $\Phi_N \equiv \Psi_{N,q_0}^{(0)}$ is one of the functions Φ_N^{S+}, $\Phi_N^{T,+}$, $\Phi_N^{T,-}$. $\mathcal{O}_{N,q_0}(Q^2)$ is the quark-gluon-antiquark operator with the lowest anomalous dimension corresponding to the wave function $\Psi_{N,q_0} = \Psi_{N,q_0}^{(0)} + N_c^{-2}\delta\Psi_{N,q_0}$ and normalized at the scale Q^2. Thus, the calculation of $1/N_c^2$ corrections to the moments of the structure

functions consists in two separate tasks. First, one has to calculate the $\mathcal{O}(1/N_c^2)$ corrections to the anomalous dimension of $\mathcal{O}_{N,q_0}$, or equivalently to the energies (22) of the lowest eigenstates and, second, calculate the mixing coefficients $\langle\Psi^{(0)}_{N,q}|\mathcal{H}^{(1)}|\Psi^{(0)}_{N,q_0}\rangle$ with higher levels; evolution of the latter can be taken into account in the leading large-N_c approximation of Refs. [10, 11]. We address both questions in what follows.

4. The $\mathcal{O}(1/N_c^2)$ correction δE_N to the energy of the lowest eigenstates is given by Eq. (26) for $q = q_0$. The calculation is most easily done by going over to the so-called conformal basis [8, 9, 10, 11]. The idea is to define a basis of (orthogonal with respect to (15)) polynomials that satisfy the conformal constraints in (12) and, in addition, diagonalize the two-particle Casimir operator (8) in one of the channels, e.g.

$$L_{12}^2\, Y^{(12)3}_{N,n}(x_1,x_2,x_3) = (n+5/2)(n+3/2)\, Y^{(12)3}_{N,n}(x_1,x_2,x_3) \tag{29}$$

and, similarly, $Y^{1(23)}_{N,n}$ and $Y^{(13)2}_{N,n}$. Here, the superscripts indicate the order in which the conformal spins of the three partons sum up to the total conformal spin $N+7/2$. The three different sets of Y-functions are related to each other through the Racah $6j$-symbols of the $SL(2)$ group. Permutation symmetry between the quarks implies that $Y^{(23)1}_{N,n}(x_1,x_2,x_3) = (-1)^n\, Y^{(12)3}_{N,n}(x_3,x_2,x_1)$. The explicit expressions for the $Y-$functions are given in terms of the Jacobi polynomials [9, 10].

The expansion coefficients of the lowest eigenstates (18) in each of the three conformal basis are easily obtained [11, 12]

$$\begin{aligned}\Phi_N^{T,\pm} &= \sum_{n=0}^{N}(-1)^n(N+n+5)\left[\frac{N+3}{n+1}\pm(n+3)\right]Y^{(12)3}_{N,n} \\ &= \sum_{n=0}^{N}\left[(-1)^{N-n}\pm 1\right](2n+3)(n+2)\frac{2(N+n)+9\pm1}{2(N-n)+3\mp1}Y^{(31)2}_{N,n}\end{aligned} \tag{30}$$

and $\Phi_N^{S+} = (\Phi_N^{T,+}+\Phi_N^{T,-})/2$. The normalization is

$$\begin{aligned}\|Y^{(12)3}_{N,n}\|^2 &= \frac{(n+1)(N-n+1)}{(n+2)(n+3)(N+n+5)}\,, \\ \|Y^{(31)2}_{N,n}\|^2 &= \frac{2(n+1)(N-n+1)(N-n+2)}{(n+2)(2n+3)(N+n+4)(N+n+5)}\,.\end{aligned} \tag{31}$$

Notice that the two-particle kernels defined in (7) become diagonal in the corresponding basis. Expanding each contribution in a suitable basis, one obtains the matrix elements, $\langle\Psi_N^{T,\pm}|\mathcal{H}_T^{(1)}|\Psi_N^{T,\pm}\rangle$ and $\langle\Psi_N^{S+}|\mathcal{H}_{S+}^{(1)}|\Psi_N^{S+}\rangle$, as finite sums over two-particle spins n. Full expressions are rather cumbersome and will be presented elsewhere [12].

Expanding the resulting expressions for δE_N in powers of $1/(N+3)$ we obtain

$$\delta E_N^S = -2\left(\ln(N+3)+\gamma_{\rm E}+\frac{3}{4}-\frac{\pi^2}{6}\right)+\mathcal{O}\left(\frac{\ln^2(N+3)}{(N+3)^2}\right)\,, \tag{32}$$

$$\delta E_N^{T,\pm} = \delta E_N^S \pm \frac{4}{N+3}\left[\left(3-\frac{\pi^2}{3}\right)(\ln(N+3)+\gamma_{\rm E})-\frac{5}{2}+\frac{\pi^2}{3}\right]+\mathcal{O}\left(\frac{\ln^2(N+3)}{(N+3)^2}\right)\,.$$

N		0	1	2	10	20	50	100
$\mathcal{E}_N^{T,+}$:	exact	6.1111	8.1111	9.5902	15.3901	18.6260	23.2234	26.8234
	Eq.(32)	6.1146¶	8.1112	9.5805	15.3779	18.6193	23.2233	26.8260
	large-N_c	6.5000	8.7500	10.4000	16.8885	20.5144	25.6717	29.7134
$\mathcal{E}_N^{T,-}$:	exact	–	11.5556	12.2111	16.3820	19.1948	23.4791	26.9596
	Eq.(32)	–	10.9640¶	11.8972	16.3258	19.1742	23.4763	26.9612
	large-N_c	–	11.7500	12.8000	17.8116	21.0362	25.8981	29.8299
$\mathcal{E}_N^{S}$:	exact	8.5556	9.7550	10.8914	15.8758	18.9033	23.3480	26.8932
	Eq.(32)	7.9794¶	9.5376	10.7389	15.8519	18.8968	23.3498	26.8936
	large-N_c	8.5000	10.2500	11.6000	17.3500	20.7753	25.7849	29.7716

Table 1: The lowest anomalous dimensions in the spectrum of twist-3 operators for different N calculated by taking into account the $\mathcal{O}(1/N_c)$ correction, Eqs. (32) and (25), in comparison with the corresponding exact numerical results and leading large-N_c expressions (22).

With this correction, Eq. (25) gives an excellent description of the lowest anomalous dimension in the spectrum of twist-3 operators for *all* integer $N \geq 0$, see Table 1. Note that to this accuracy $\mathcal{E}_N^{T,+} + \mathcal{E}_N^{T,-} = 2\mathcal{E}_N^{S}$, cf. Eq. (23).

The following comments are in order.

First, we note that δE_N has the same large-N behavior, $\delta E_N \sim \ln N$, as the leading large-N_c result in (22). The coefficient in front of the $\ln N$ term at large N is redefined, therefore, from $2N_c$ to $4C_F = 2(N_c^2-1)/N_c$, in agreement with [2, 3]. It is possible to show [13] that this coefficient is exact and higher order $1/N_c^2$ corrections do not grow with N, $\mathcal{E}_{N,q_0} = 2C_F \ln N + \mathcal{O}(N^0)$.

Second, the constant $\mathcal{O}(N^0)$ term in the large N expansion of the anomalous dimensions (32) does not agree with [2, 3, 4]. The difference is due to the contribution of the operators $V_{qg}^{(1)}$ and $U_{qg}^{(1)}$ in (6) that was overlooked in the previous works. The result for the $\mathcal{O}(1/N)$ correction is new.

Finally, from the expressions in (32) it is easy to read out the corresponding modification of the DGLAP splitting functions (2), $P_f(z) = P_f^{(0)}(z) + P_f^{(1)}(z)$. For example

$$P_{g_2}^{(1)}(z) \overset{z\to 1}{=} \frac{2}{N_c}\left\{\left[\frac{-1}{1-z}\right]_+ + \left(\frac{3}{4}-\frac{\pi^2}{6}\right)\delta(1-z) + \frac{1}{2} + \mathcal{O}(1-z)\right\}. \tag{33}$$

This result is exact to the $\mathcal{O}(1/N_c^4)$ accuracy and neglecting all terms that vanish at $z \to 1$. Expressions for $P_e^{(1)}(z)$ and $P_{h_L}^{(1)}(z)$ have similar structure.

The opposite limit of the small$-z$ behavior of the DGLAP splitting functions $P_f(z)$ is of special interest. It is well known that this behavior is governed by singularities of the

¶The difference of these values with the exact result is entirely due to the truncation of the $1/(N+3)-$expansion in (32). Since only one independent operator exists in these three cases, there is no mixing and the $\mathcal{O}(1/N_c)$ approximation to the energies $\mathcal{E}_{N=0}^{S}$, $\mathcal{E}_{N=0}^{T,+}$ and $\mathcal{E}_{N=1}^{T,-}$ is in fact exact.

anomalous dimensions $\mathcal{E}_N$ in the complex N-plane. In particular, the leading large-N_c anomalous dimensions (22) have simple poles at negative integer $N \leq -3$. The pole at $N = -3$ corresponds to $P^{(0)}(z) \overset{z\to 0}{\sim} z^0$ in (2). With this connection in mind, we have studied the analytic structure of $1/N_c^2$ corrections to the anomalous dimensions, δE_N, continued analytically to the complex N plane. The calculation is straightforward, albeit tedious [12]. We find that the analytic continuation has to be performed separately for even and odd N, as familiar from studies of the evolution of twist-2 parton distribution beyond the leading order [14]. Similarly to the latter case, we define[‖]

$$\int_{-1}^{1} dx\, x^N\, f(x) = \frac{1+(-1)^N}{2} \int_0^1 dx\, x^N\, f_{\rm even}(x) + \frac{1-(-1)^N}{2} \int_0^1 dx\, x^N\, f_{\rm odd}(x) \qquad (34)$$

and consider the evolution of even- and odd-signature component of $f = e\,,\widetilde{h}_L\,,\widetilde{g}_2$ separately. When the restriction to even (odd) N is imposed, the normalized matrix elements $\langle\Phi_N|\mathcal{H}^{(1)}|\Phi_N\rangle/\|\Phi_N\|^2$ develop singularities at, generically, $N = 0, -1, -2$. With one exception, all singularities to the right of $N = -3$ are simple poles and appear due to vanishing of the norm of the leading large-N_c wave functions $\|\Phi_N\|^2$. They give rise to small-z behavior of the $1/N_c^2$ correction to the DGLAP splitting functions of the form

$$\begin{aligned}
P^{(1)}_{g_2^{\rm even}}(z) &\overset{z\to 0}{=} \frac{1}{N_c}\left\{ -5\,\frac{17-2\pi^2}{51-4\pi^2}\frac{1}{z^2} + 6 + \mathcal{O}(z)\right\} ,\\
P^{(1)}_{g_2^{\rm odd}}(z) &\overset{z\to 0}{=} \frac{1}{N_c}\left\{ \frac{6}{6-\pi^2}\frac{\ln z}{z} + \frac{\pi^4 - 18\pi^2 + 54 + 72\,\zeta(3)}{2(6-\pi^2)^2}\,\frac{1}{z} + \mathcal{O}(z)\right\} .
\end{aligned} \qquad (35)$$

This asymptotics is more singular than that of $P^{(0)}_{g_2}(z)$ and is in apparent contradiction with the Regge theory expectations. The origin of these "spurious" singularities and their physical significance is not clear and deserves a special study that goes beyond the tasks of this letter. We would like to stress that appearance of singularities of the anomalous dimensions to the right from $N = -3$ can well be artifact of the $1/N_c$ expansion, since close to the singularities the $1/N_c^2$ correction dominates over leading order term in (25) and the $1/N_c^2$ expansion formally breaks down.

Another delicate point is that analytic continuation in N has to be done in (28) simultaneously with the analytic continuation of the sum over anomalous dimensions $\mathcal{E}_{N,q_\ell}$ belonging to different "trajectories" parameterized by integer ℓ [11]. To the best of our knowledge, the problem of analytic continuation from a set of discrete points $\mathcal{E}_{N,q}$ on a $(N,q)-$plane has never been addressed in mathematical literature.

To illustrate that the problem is nontrivial, consider the trace of the (full) Hamiltonian (5) over the subspace spanned by wave functions with given $N \geq 0$

$$\mathrm{Tr}_N\, \mathcal{H} = \sum_{\ell=0}^{N} \mathcal{E}_{N,q_\ell}\,. \qquad (36)$$

[‖]This corresponds to the decomposition over partial waves with definite signature.

It is equal to the sum of all $N+1$ anomalous dimensions and is easily calculated *exactly* as a sum of diagonal matrix elements in any suitable basis. We obtain, in particular

$$
\begin{aligned}
\mathrm{Tr}_N\,\mathcal{H}^{(0)}_{S^+} &= 2(2N+5)\Big[\psi(N+3)+\gamma_{\rm E}\Big] - \frac{11}{2}N - 13 + \frac{1}{N+2} + \frac{1}{N+3}\,, \\
\mathrm{Tr}_N\,\mathcal{H}^{(1)}_{S^+} &= (N+2)\Big[\psi(N+3)+\gamma_{\rm E}\Big] - \frac{7}{4}N - 5 + \frac{1}{2(N+2)} + \frac{5}{2}\ln 2 \\
&\qquad + (-1)^N\left\{\frac{5}{4}\left[\psi\left(\frac{N}{2}+\frac{3}{2}\right) - \psi\left(\frac{N}{2}+2\right)\right] + \frac{1}{2(N+2)} + \frac{1}{2(N+3)}\right\}.
\end{aligned}
\tag{37}
$$

Notice now that the trace of already the leading large-N_c Hamiltonian $\mathcal{H}^{(0)}$ is singular at $N=-2$. This implies that either one (or several) trajectories of the anomalous dimensions, $\mathcal{E}_{N,q_\ell}$, becomes singular at this point, or the (analytically continued) sum over the trajectories in (28) diverges. In both cases, analytic continuation of the lowest trajectory (22) beyond $N=-2$ is questionable and, therefore, asymptotic expressions in (35) (or, equivalently, the DGLAP-evolution for $z < 1/N_c^2$) have to be taken with caution.

5. According to (28), the moments of twist-3 distributions receive contributions from the whole tower of operators with non-minimal anomalous dimensions at the level of $\mathcal{O}(1/N_c^2)$ corrections. Aside from the "energy denominators", the coefficients in front of these operators in (28) are given by (normalized) matrix elements $\langle \ell'|\mathcal{H}^{(1)}|\ell=0\rangle$ of the perturbation $\mathcal{H}^{(1)}$

$$
\langle \ell'|\mathcal{H}^{(1)}|\ell\rangle = \frac{\langle \Psi^{(0)}_{N,q_{\ell'}}|\mathcal{H}^{(1)}|\Psi^{(0)}_{N,q_\ell}\rangle}{\|\Psi^{(0)}_{N,q_{\ell'}}\|\,\|\Psi^{(0)}_{N,q_\ell}\|},
\tag{38}
$$

where $\ell,\ell' = 0,1,\ldots,N$ numerate the anomalous dimensions from below. Our main observation is that the matrix elements (38) are concentrated close to the diagonal $\ell=\ell'$, see Figs. 1, 2 and Table 2. In addition, the lowest anomalous dimensions $E_{N,q_0}\equiv E_N$ (22) are separated from the rest of the spectrum by a finite gap $E_{N,q_1}-E_N = 0.227$ as $N\to\infty$ [8, 11]. These two properties guarantee that the expansion in (28) is rapidly converging and remains well defined in the large-N limit (when the number of eigenstates diverges). The general structure for the cases of $\mathcal{H}_{S^+}$ and $\mathcal{H}_T$ is very similar; we present the results for S^+ as related to the structure function $g_2(x)$ and therefore being of more direct phenomenological significance.

At large N the matrix elements $\langle \ell|\mathcal{H}^{(1)}|0\rangle$ can be calculated semi-classically using the approach developed in [11]. It turns out that the matrix elements $|\langle \ell|\mathcal{H}^{(1)}|0\rangle|$ approach maximal value $\sim 1/\sqrt{\ln N}$ at $\ell\sim\ln N$, while for $\ell\gg\ln N$ they rapidly decrease as $1/\ell^2$. As the result, the sum in (28) is effectively cut off at $\ell\sim\ln N$ terms. For example, for $N\sim 100$ only ~ 5 first (lowest) levels give a sizeable contribution to the r.h.s. of (28), cf. Fig. 2. Analytic expressions for the matrix elements $\langle \ell|\mathcal{H}^{(1)}|0\rangle$ in the large N limit are complicated and will be given elsewhere [12].

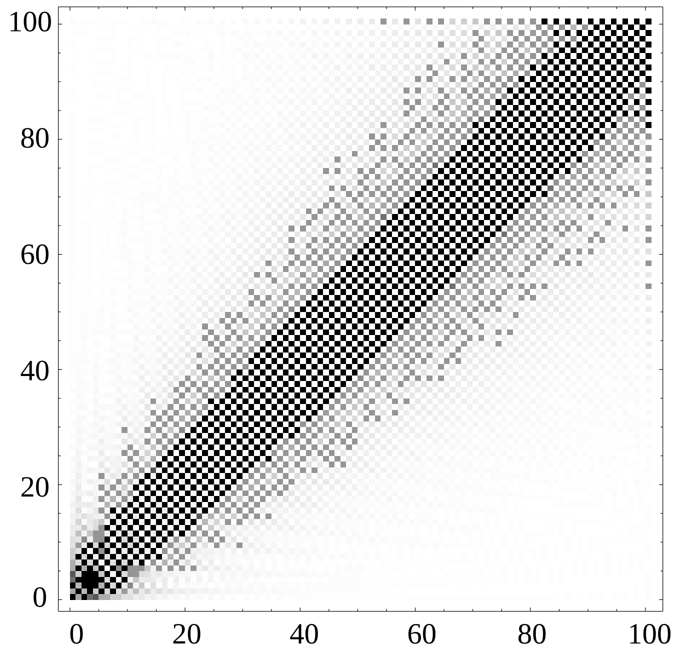

Figure 1: The density plot of the mixing matrix $|\langle\ell'|\mathcal{H}^{(1)}_{S+}|\ell\rangle|$ at $N = 100$; $\ell = 0$ corresponds to the lowest eigenstate Ψ^{S+}_N of the large-N_c Hamiltonian $\mathcal{H}^{(0)}_{S+}$. Notice "chess-board" structure with alternating large (dark) and small (light) elements.

An independent argument in favor of the dominance of the operators with lowest anomalous dimensions comes from the structure of matrix elements $\langle\!\langle \mathcal{O}_{N,q}\rangle\!\rangle$ in the large-N limit. In an analogy with twist-2 distributions, we can write the matrix elements $\langle\!\langle \mathcal{O}_{N,q}\rangle\!\rangle$ as (generalized) moments of nonperturbative (chiral-odd or chiral-even) distribution function $D_f(x_1, x_2, x_3)$ describing quark-gluon-antiquark correlations in the nucleon, schematically**

$$\langle\!\langle \mathcal{O}_{N,q}\rangle\!\rangle = \int_{-1}^{1}[\,dx\,]\,\Psi_{N,q}(x_1,x_2,x_3)\,D_f(x_1,x_2,x_3), \tag{39}$$

where $[dx] = dx_1\,dx_2\,dx_3\,\delta(x_1 + x_2 + x_3)$ and the variables x_i have the meaning of the quark, antiquark and gluon momentum fractions, $-1 \le x_i \le 1$.

At large N, the eigenfunctions $\Psi_{N,q}(x_i)$ are sharply peaked at the boundary of the integration region $x_1 + x_2 + x_3 = 0$ so that the matrix elements (39) probe the behavior of the quark-gluon distribution functions $D(x_i)$ near the kinematical boundaries. The precise position of the peak depends on q. It turns out that the eigenfunctions of the low-lying states are peaked at $x_1 = -x_3 = \pm 1,\ x_2 = 0$ corresponding to configurations when the quark and the antiquark carry all the momentum of nucleon and the gluon is soft. For higher states the position of the peak is shifted gradually to $2x_1 = 2x_3 = -x_2 = \pm 1$ corresponding to a hard gluon and (relatively) soft quark and antiquark. If one assumes that the nonperturbative distributions are generated by gluon radiation already at low

**see Appendix A in [11] for exact expressions.

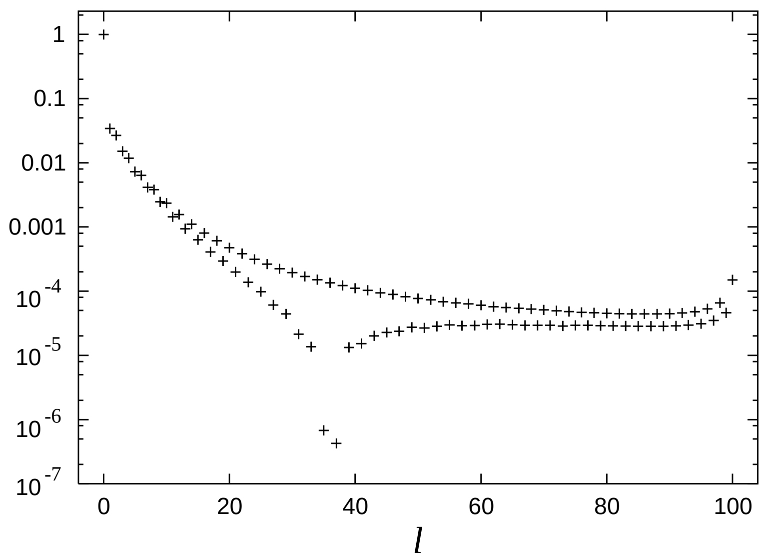

Figure 2: Exact numerical results for the coefficients $\frac{|\langle\Psi_{N,q_\ell}|\Phi_N^S\rangle|}{\|\Psi_{N,q_\ell}\|\|\Phi_N^S\|}$ that enter the expansion (17) of the $N = 100$th moment of $\widetilde{g}_2(x, Q^2)$ in contributions of multiplicatively renormalized operators. Notice smallness of all coefficients with $\ell \geq 1$ and their different behavior for even and odd ℓ.

scales, the configurations with soft gluons should be enhanced and those with hard gluons suppressed. We conclude that the contribution of the conformal operators with large anomalous dimensions to the moments of the structure functions, (28), is suppressed because of smallness of both *perturbative* mixing coefficients and the corresponding *non-perturbative* matrix elements as describing rare parton configurations in nucleon involving a hard gluon.

To summarize, we have argued that (up to $\mathcal{O}(1/N_c^4)$ corrections) the contribution of multiplicatively renormalizable operators with higher anomalous dimensions to the moments of twist-3 structure functions $e(x)$, $\widetilde{h}_L$ and $\widetilde{g}_2(x)$ is small as compared with that of the ground state operator. With the increasing precision of the experimental data this contribution can be estimated in a "two-channel" approximation in which one supplements the dominant ground state quark-gluon component with one extra effective partonic component in order to account for the admixture of $\sim \ln N$ lowest eigenstates.

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ℓ	0	1	2	3	4	5	6
$N=0$	1.0000						
$N=1$	0.9996	0.0283					
$N=2$	0.9991	0.0080	0.0412				
$N=3$	0.9995	0.0037	0.0295	0.0063			
$N=4$	0.9994	0.0030	0.0305	0.0001	0.0146		
$N=5$	0.9995	0.0056	0.0278	0.0005	0.0096	0.0037	
$N=6$	0.9995	0.0093	0.0277	0.0010	0.0108	0.0008	0.0078

Table 2: Exact numerical results for the coefficients $\frac{|\langle\Psi_{N,q_\ell}|\Phi_N^S\rangle|}{\|\Psi_{N,q_\ell}\|\|\Phi_N^S\|}$ that enter the expansion (17) of the lowest moments of $\widetilde{g}_2(x,Q^2)$ in contributions of multiplicatively renormalized operators. Notice dominance of the coefficients with even ℓ over those with odd ℓ for a given N.

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